Section 2.4

Grade: «grade»

Subject: «subject»

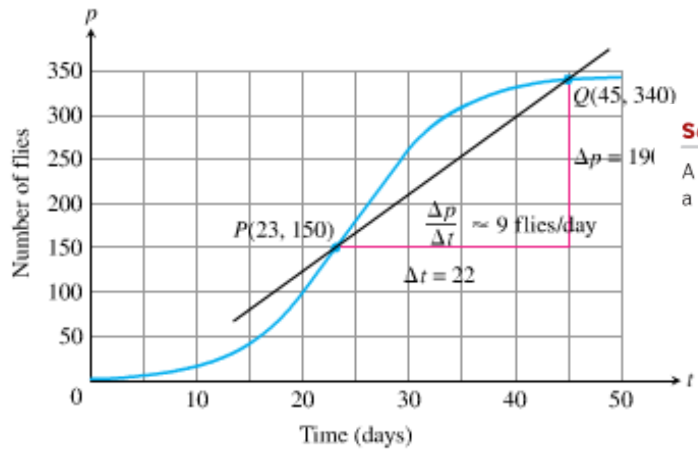
Date: «date»

Lab: Zooming In

EXAMPLE 1 Finding Average Rate of Change

Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

Video



Secant to a Curve

A line through two points on a curve is a **secant to the curve.**

Figure 2.27 Growth of a fruit fly population in a controlled experiment.
Source: Elements of Mathematical Biology. (Example 2)

EXAMPLE 2 Growing Drosophila in a Laboratory

Use the points $P(23, 150)$ and $Q(45, 340)$ in Figure 2.27 to compute the average rate of change and the slope of the secant line PQ.

As suggested by Example 2, *we can always think of an average rate of change as the slope of a secant line.*

Secant-tangent of curve.gsp

Q	slope of line PQ

Tangent to a Curve

1. We start with what we can calculate, namely, the slope of a secant through P and a point Q nearby on the curve.

2. We find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.

3. We define the *slope of the curve* at P to be this number and define the *tangent to the curve* at P to be the line through P with this slope.

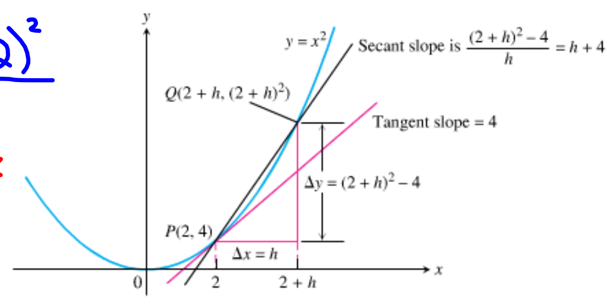
EXAMPLE 3 Finding Slope and Tangent Line

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Y_1 of 1st pt $\quad Y_1(2)$

$$\lim_{h\to 0} \frac{(2+h)^2 - (2)^2}{h}$$

Δx



$$\lim_{h\to 0} \frac{\cancel{4} + 4h + h^2 \cancel{-4}}{h}$$

$$\lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$\lim_{h\to 0} 4+h = 4 + (0) = 4$$

Slope of tangent at x=2

Slope of curve at x=2

instantaneous rate at x=2

DEFINITION Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

EXAMPLE 4 Exploring Slope and Tangent

Let $f(x) = 1/x$.

(a) Find the slope of the curve at $x = a$.
(b) Where does the slope equal $-1/4$?
(c) What happens to the tangent to the curve at the point $(a, 1/a)$ for different values of a?

The expression

$$\frac{f(a+h) - f(a)}{h}$$

is the **difference quotient of f at a.**

a) $\lim_{h \to 0} \dfrac{\boxed{\frac{1}{a+h} - \frac{1}{a}}}{h}$

$= \lim_{h \to 0} \dfrac{\boxed{\frac{a-(a+h)}{a(a+h)}}}{h}$

$= \lim_{h \to 0} \dfrac{a-(a+h)}{a(a+h)} \cdot \dfrac{1}{h}$

$= \lim_{h \to 0} \dfrac{\cancel{a-a}-\cancel{h}}{a(a+h)} \cdot \dfrac{1}{\cancel{h}}$

$= \lim_{h \to 0} \dfrac{-1}{a(a+h)}$

$= \dfrac{-1}{a(a+0)} = \boxed{-\dfrac{1}{a^2}}$

b) $-\dfrac{1}{a^2} = -\dfrac{1}{4}$

$a = \pm 2$

when $x=2$ and $x=-2$

1 Find the slope at the point P(2,4) of

$y = x^2$

Normal to a Curve

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

EXAMPLE 5 Finding a Normal Line

Write an equation for the normal to the curve $f(x) = 4 - x^2$ at $x = 1$.

$$\lim_{x \to 0} \frac{4-(1+h)^2-(4-1)}{h}$$

$$= \lim_{h \to 0} \frac{4-(1+2h+h^2)-3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3}-2h-h^2\cancel{-3}}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}(2+h)}{\cancel{h}} = -2+0 = -2 \quad \text{(Slope of the tangent)}$$

Slope of the normal: $\frac{1}{2}$ point $(1,3)$

$$\boxed{y-3 = \frac{1}{2}(x-1)}$$

2 Answer?

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

$f(t) = 16t^2$

EXAMPLE 6 Investigating Free Fall

Find the speed of the falling rock in Example 1, Section 2.1, at $t = 1$ sec.

$$\lim_{h \to 0} \frac{16(1+h)^2 - 16(1)^2}{h}$$

$$= \lim_{h \to 0} \frac{16(1 + 2h + h^2) - 16}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{16} + 32h + 16h^2 \cancel{- 16}}{h}$$

$$= \lim_{h \to 0} \frac{16\cancel{h}(2+h)}{\cancel{h}}$$

$= 16(2+0) = 32$ ft/sec

speed of rock at $t = 1$ sec.